A CHARACTERIZATION OF 3-SPACE BY PARTITIONINGS

BY

R. H. BING

While it is apparent that any Euclidean n-sphere can be partitioned, it was shown only recently that any compact locally connected metric continuum is partitionable(1). In this paper it is shown how a 3-sphere can be characterized in terms of its partitionings.

Definitions and notation. We designate *Euclidean n-space* by E^n. The *unit n-sphere with center at the origin* in E^{n+1} is designated by S^n. A continuum is called a *simple closed curve*, *simple surface* (or *2-sphere*), or *3-sphere* according as it is topologically equivalent to S^1, S^2, or S^3. While a 3-sphere may be regarded as the set of points in E^4 with coordinates (x, y, z, w) satisfying $x^2+y^2+z^2+w^2=1$, we prefer to think of it as E^3 plus a point added in such a fashion that the exterior of a cube is topologically equivalent to its interior.

A simple surface C in E^3 is called *tame* if there is a homeomorphism of E^3 into itself that carries C into S^2. If there is no such homeomorphism, C is called *wild*.

We shall suppose that *space* S is metric, compact, locally connected, and connected.

A *partitioning* of S is a collection of mutually exclusive open sets whose sum is dense in S. A sequence $G_1, G_2, \cdots$ of partitionings is a *decreasing sequence of partitionings* if G_{i+1} is a refinement of G_i and the maximum of the diameters of the elements of G_i approaches 0 as i increases without limit. A partitioning is *regular* if each of its elements is the interior of the closure of this element.

A regular partitioning G is a *brick* partitioning if each element of G is uniformly locally connected and the interior of the closure of the sum of each pair of elements of G is uniformly locally connected. We know that *S has a decreasing sequence of brick partitionings*(2).

The *boundary* of a set A will be denoted by $F(A)$.

Characterization of a 3-sphere. A 3-sphere may be partitioned in many ways. The boundaries of some elements of a partitioning may be simple surfaces, the boundaries of others may be tori, and those of some may not even

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(1) R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1101–1110, and E. E. Moise, *Grille decomposition and convexification theorems for compact continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

(2) R. H. Bing, *Complements of continuous curves*, Fund. Math. vol. 36 (1950) pp. 303–318.

be locally simply connected. However, a 3-sphere has a partitioning in which all the elements of the partitioning have simple surfaces for boundaries. The following theorem gives a characterization of a 3-sphere in terms of the manner in which it may be partitioned.

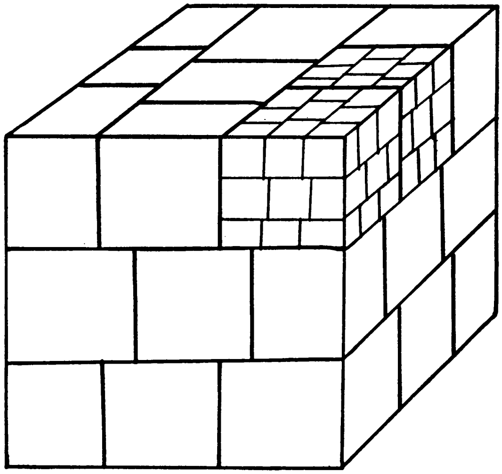

FIG. 1

THEOREM 1. *A necessary and sufficient condition that S be a 3-sphere is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties*:

(1.1) *The boundary of each element of G_i is a simple surface.*

(1.2) *If the boundaries of two elements of $\sum G_i$ intersect, this intersection is a 2-cell.*

(1.3) *The intersection of the boundaries of three elements of G_i is one-dimensional at each of its points.*

(1.4) *If g is an element of G_{i-1} ($g=S$ if $i=1$) the elements G_i in g may be ordered $g_1, g_2, \cdots, g_n$ so that $F(g_j)$ [$j=1$ (if $i>1$), $2, \cdots, n$] intersects $F(g)+F(g_1)+\cdots+F(g_{j-1})$ in a connected set.*

To see that the condition is necessary, one may consider the ways in which a cube may be filled with oriented rectangular solids all the same height. See Figure 1. Before showing that the condition is sufficient, we develop some other theorems to use. In Theorem 3 we show that there is a sequence of partitionings of S^3 that corresponds to $G_1, G_2, \cdots$. Using Theorem 5 we show further that there is such a sequence of partitionings of S^3 which is decreasing. The proof of the sufficiency in Theorem 1 follows Theorem 6.

The following is a combinatorial result. Although it will not be used in proving Theorem 1, it may be of interest for its own sake.

THEOREM 2. *If C is a finite complex which is a 3-manifold without boundary, then C is a 3-sphere if the 3-simplexes of C may be ordered $c_1, c_2, \cdots, c_n$ such that c_j $(j=2, \cdots, n)$ intersects $c_1+c_2+\cdots+c_{j-1}$ in a connected set.*

Proof. Consider c_i to be a regular tetrahedron whose edges are of length 1. Let c_{ij} $(j=1, 2, 3, 4)$ be a regular tetrahedron in c_i such that c_{ij} has a vertex

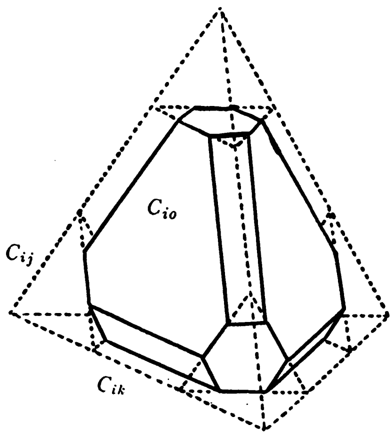

FIG. 2

in common with a vertex of c_i and the edges of c_{ij} are of length 1/3. Let c_{ik} $(k=5, 6, \cdots, 10)$ be a 5-sided polyhedron in c_i such that two faces of c_{ik} are triangles which lie in faces of different c_{ij}'s, two other faces are trapezoids of heights 1/9 which lie in faces of c_i, and the fifth face is a rectangle whose interior lies in the interior of c_i. Let c_{i0} be the 14-sided polyhedron equal to the closure of $c_i-(c_{i1}+c_{i2}+\cdots+c_{i10})$. See Figure 2.

Denote by G the regular partitioning of C where the closures of the elements of G are of three types: (1) a c_{i0}; (2) if p is a vertex of a 3-simplex in C, the sum of all c_{ij}'s containing p; and (3) if e is an edge of a 3-simplex in C, the sum of all c_{ik}'s having the middle third of e as an edge.

Each element of G is an open 3-cell whose boundary is a simple surface. To establish that elements of types 2 and 3 have these properties, one uses the hypothesis that C is a 3-manifold([3]). If the boundaries of two elements of G intersect, the intersection is a 2-cell. If the boundaries of three elements of G intersect, the intersection is an arc. Furthermore, the elements of G may be ordered $g_1, g_2, \cdots, g_m$ so that $F(g_j)$ intersects $F(g_1)+F(g_2)+\cdots+F(g_{j-1})$ in a connected set. To get such an ordering, suppose that g' precedes g'' if the first element of $c_{1\,0}, c_{1\,1}, \cdots, c_{1\,10}, c_{2\,0}, \cdots, c_{n\,10}$ which is in $\bar{g}'+\bar{g}''$ is in $\bar{g}'$. Hence it follows from Theorem 3 that C is a 3-sphere.

THEOREM 3. *Suppose T is a homeomorphism carrying a simple surface F_0*

([3]) H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, New York, 1947, Theorem 1, p. 205.

in S into a tame simple surface C in E^3, and G is a regular partitioning of S satisfying the following conditions:

(3.1) *The boundary of each element of G is a simple surface.*

(3.2) *If the boundaries of two elements of G intersect, their intersection is the sum of a finite number of mutually separated 2-cells.*

(3.3) *The intersection of the boundaries of three elements of G is one-dimensional at each of its points.*

(3.4) *F_0 is the boundary of the closure of the sum of an ordered subcollection $[g_1, g_2, \cdots, g_n]$ of G such that $F(g_j)$ $(j=1, 2, \cdots, n)$ intersects $F_0+F(g_1)+\cdots+F(g_{j-1})$ in a connected set.*

Then there is a partitioning $[h_0, h_1, \cdots, h_n]$ of E^3 and a homeomorphism T' of the boundary (in S) of $g_1+g_2+\cdots+g_n$ into the boundary (in E^3) of $h_1+h_2+\cdots+h_n$ such that h_0 is the exterior of C, $F(h_i)$ is a tame simple surface, $T=T'$ on F_0, and $T'[F(g_i)]=F(h_i)$.

Proof. We shall use induction on n. If $n=1$, the theorem is true because we can let h_1 be the interior of C and T' be T.

If p is an interior point of a 2-cell K (interior with respect to K) which lies in the intersection of the boundaries of two elements g_i, g_j of G, p is not on the boundary of any other element g_r of G or else $F(g_i)\cdot F(g_r)$ would contain a 2-cell intersecting p. But this is impossible since both K and $F(g_i)\cdot F(g_r)$ lie on the simple surface $F(g_i)$, and $F(g_r)\cdot K$ is prevented by condition (3.3) from being two-dimensional. Hence, the common part of the boundaries of three elements of G is a subset of the boundaries of the sum of the 2-cells which is the common part of some two of them. Hence, condition (3.3) may be replaced by the following condition.

(3.3)′ *if the boundaries of three elements of G intersect, their intersection is the sum of a finite number of arcs.*

We shall show that $\bar{g}_1+\bar{g}_2+\cdots+\bar{g}_n-F_0$ is connected. If it were the sum of two mutually separated sets H and K, F_0 would be the boundary of $H+K$ and each of the sets $\overline{H}$, $\overline{K}$ would intersect F_0. Then $\overline{H}\cdot\overline{K}$ would contain a point p of F_0. There would be elements g_i and g_j of $[g_1, g_2, \cdots, g_n]$ in H and K respectively such that $\bar{g}_i\cdot\bar{g}_j$ contains p. It follows from (3.3) that $F_0\cdot F(g_i)\cdot F(g_j)$ is one-dimensional and from (3.2) that $F(g_i)\cdot F(g_j)$ contains a 2-cell. Hence H and K could not be mutually separated because the points of $\bar{g}_i\cdot\bar{g}_j$ not in F_0 would be limit points of H and K.

We now show that if $n\geqq 2$, then $F_0\cdot F(g_1)$ is topologically equivalent to a bounded subset of the plane whose boundary is the sum of a finite number of mutually exclusive circles. Since $F(g_1)$ is a simple surface which contains a point not of F_0, $F_0\cdot F(g_1)$ is a proper subset of F_0. From condition (3.4) we find that $F_0\cdot F(g_1)$ is connected. Furthermore, it follows from condition (3.2) that $F_0\cdot F(g_1)$ is the sum of a finite number of 2-cells each of which is a component of the intersection of $F(g_1)$ and the boundary of an element of G not in $[g_1, g_2, \cdots, g_n]$. As a result of condition (3.3′), if two of these 2-cells intersect, their common part is the sum of a finite number of arcs. Hence, $F_0 \cdot F(g_1)$ is topologically equivalent to a bounded subset of the plane whose boundary is the sum of a finite collection of mutually exclusive circles.

Let $J_1, J_2, \cdots, J_t$ be the components of the boundary with respect to F_0 of $F_0 \cdot F(g_1)$, I_i $(i=1, 2, \cdots, t)$ be the component of $F_0 - J_i$ containing no point of $F(g_1)$, and K_i be the component of $F(g_1) - J_i$ containing no point of F_0. The boundary of $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_n$ is $\sum(I_i + J_i + K_i)$ as can be shown from conditions (3.1), (3.2), and (3.3). However, no component of $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_n$ intersects two elements of $\{I_i + J_i + K_i\}$. If this were not true there would be an integer j such that $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_j$ contains a continuum intersecting two elements of $\{I_i + J_i + K_i\}$ but $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_{j-1}$ contains no such continuum. However, $F(g_j)$ would not intersect $F_0 + F(g_1) + \cdots + F(g_{j-1})$ in a connected set as required by condition (3.4). Hence, each component of $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_n$ has one of the simple surfaces of $\{I_i + J_i + K_i\}$ for a boundary.

If H_i $(i=1, 2)$ is a collection of t mutually exclusive simple closed curves on a simple surface such that no element of H_i separates any other two elements of H_i from each other on the surface, there is a homeomorphism of the surface into itself that carries the elements of H_1 into the elements of H_2. Also, for each homeomorphism T'' of the surface of a cube in E^3 into itself, there is a homeomorphism of E^3 into itself that preserves T'' on the surface of the cube. Hence, if T is the transformation mentioned in the statement of Theorem 3, without loss of generality we may suppose that C is the surface of a cube and the $T(J_1), T(J_2), \cdots, T(J_t)$ are circles $C_1, C_2, \cdots, C_t$ all lying in one base of C. Let W_i $(i=1, 2, \cdots, t)$ be the hemispherical surface which contains C_i and lies in C plus its interior. There is a homeomorphism T_i of K_i into W_i such that if p is a point of J_i, $T(p) = T_i(p)$. Suppose $T'(p) = T(p)$ if p is a point of F_0 and $T'(p) = T_i(p)$ if p is a point of K_i. Then T' is a homeomorphism carrying $F_0 + F(g_1)$ into $C + \sum W_i$.

Each bounded complementary domain of $C + \sum W_i$ is a 3-cell and the boundary of each of these domains is tame and the image under T' of either $F(g_1)$ or the boundary of a component of $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_n$. By induction on n, the complementary domains of $C + \sum W_i$ can be partitioned and the homeomorphism T' extended on the boundaries of these partitionings so that a partitioning $[h_0, h_1, \cdots, h_n]$ and a homeomorphism T' are obtained which satisfy the conditions of the theorem.

THEOREM 4. *Suppose C is a tame simple surface in E^3 and $G_1, G_2, \cdots$ is a sequence of partitionings of E^3 satisfying the following conditions*:

(4.1) *The boundary of each element of G_i is a tame simple surface.*

(4.2) *If g is an element of G_i whose closure intersects C, $F(g) \cdot C$ is connected and does not separate $F(g)$.*

(4.3) *Each G_{i+1} is a refinement of G_i.*

(4.4) *One element of* G_i *is the exterior of* C.

(4.5) *For each positive number* ϵ *there is a positive integer* $n(\epsilon)$ *such that the closure of no element of* G_n *other than the exterior of* C *contains two points which belong to the boundary of the sum of the elements of* G_1 *and are farther apart than* ϵ.

Let K *be a closed set and* R *be a closed proper subset of* C *such that if* g *is an element of* G_1 *other than the exterior of* C, *then* $F(g) \cdot C$ *does not intersect both* R *and* K. *Then there are an integer* m *and a homomeomorphism* T *of* E^3 *into itself satisfying the following conditions*:

(4.6) *Each point of the boundary of an element of* G_1 *is invariant under* T.

(4.7) *If* g *is an element of* G_m *other than the exterior of* C, $T(\bar{g})$ *does not intersect both* R *and* K.

Proof. Let $g_1, g_2, \cdots, g_j$ be the elements of G_1 that lie on the interior of C and whose boundaries intersect R. Let J_i be a simple closed curve on $F(g_i)$ such that one complementary domain I_i of J_i in $F(g_i)$ contains $F(g_i) \cdot R$ but $I_i + J_i$ does not contain a point of K. It follows from condition (4.5) that there is an integer m such that if g is an element of G_m other than the exterior of C, the intersection of $\bar{g}$ with the boundary of the sum of the elements of G_1 is a subset of $I_1 + I_2 + \cdots + I_j$ if it intersects R.

There is a homeomorphism T_i carrying $\bar{g}_i$ into itself that leaves each point of $F(g_i)$ invariant and such that if g is an element of G_m in g_i and $\bar{g}$ intersects R, then $T_i(\bar{g})$ does not intersect K. (To see that this is so, think of $\bar{g}_i$ as a cube plus its interior whose base is $I_i + J_i$. Then T_i is a homeomorphism that moves each point of g_i toward the base.) Then the required transformation T satisfying conditions (4.6) and (4.7) is one that is the identity except on $g_1 + g_2 + \cdots + g_j$ and is T_i on g_i.

Theorem 4 could be improved by replacing G_i by G_1 in conditions (4.1), (4.2), and (4.3) but the stronger conditions as stated are needed in Theorem 5.

THEOREM 5. *Suppose that* C *is a tame simple surface in* E^3 *and* $G_1, G_2, \cdots$ *is a sequence of partitionings of* E^3 *satisfying conditions* (4.1), (4.2), (4.3), *and* (4.4) *as well as the following*:

(5.1) *For each positive integer* j *and each positive number* ϵ *there is a positive integer* $n(j, \epsilon)$ *such that the closure of no element of* G_n *other than the exterior of* C *contains two points of the boundary of the sum of the elements of* G_j *which are farther apart than* ϵ.

Then for each positive number δ, *there is an integer* m *and a homeomorphism* T *of* E^3 *into itself such that* T *leaves each point of the exterior of* C *invariant and* T *carries each other element of* G_m *into a set of diameter less than* δ.

We could not substitute for condition (4.2) one like condition (3.2) of Theorem 3.

If we tried to make such a substitution, we could not take care of the case where two mutually exclusive 2-cells D_1 and D_2 on C are "hooked together" by two elements of G_1, by four elements of G_2, by eight elements of G_3, $\cdots$ as illustrated in Figure 3. For the sake of clarity, only the elements of G_1 which "hook" D_1 and D_2 together are shown in the diagram but if g_1 and g_2 are these two elements, one may see how the other elements of G_1 can be placed by considering a homeomorphism carrying C plus its interior, $\bar{g}_1$, and $\bar{g}_2$ into the oriented rectangular solids with opposite vertices at (0, 0, 0) and (1, 1, 1), (2/5, 0, 1/5) and (3/5, 1, 2/5), and (2/5, 0, 3/5) and (3/5 ,1, 4/5) respectively.

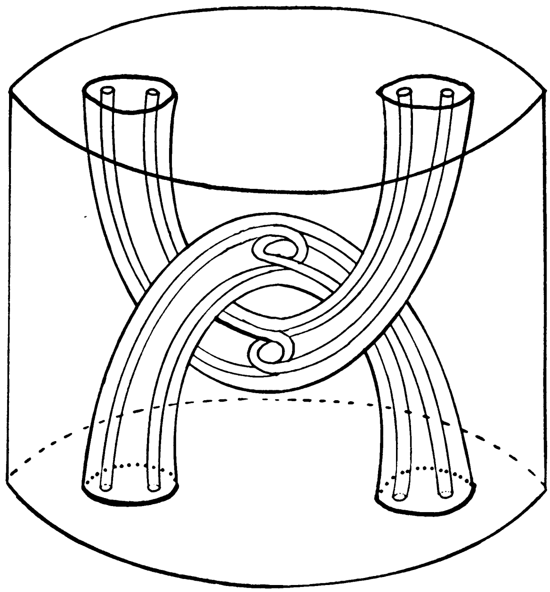

FIG. 3

Proof of Theorem 5. For convenience we suppose that C is the surface of a cube oriented with the x, y, and z axes. (If it is not one, there is a homeomorphism T_0 of E^3 into itself that carries it into one. If the elements of G_i are replaced by their images, conditions (4.1), (4.2), (4.3), (4.4), and (5.1) still hold. Also, there is a positive number δ' such that if p and q are two points of $T_0(C+$its interior) whose distance apart is less than δ', then $T_0^{-1}(p)$ and $T_0^{-1}(q)$ are no farther apart than δ.)

Let n be an integer so large that if g is an element of G_n other than the exterior of C, $F(g)$ does not intersect C in two points that are farther apart than $\delta/12$. Let S_1 be a single-valued surface defined above the base of C such that each point of $S_1\cdot C$ is $\delta/4$ units above the base of C.

We find from Theorem 4 that there is an integer n_2 and a homeomorphism T_2 of E^3 into itself such that (1) T_2 leaves invariant each point of the boundary of the sum of the elements of G_n and (2) if g is a bounded element of G_{n_2}, $T_2(\bar{g})$ does not intersect both S_1 and a point of C whose distance is as much as $2\delta/4$ from the base of C. In this application of Theorem 4 we let K and R in the statement of Theorem 4 be S_1 and the points of C which are no nearer the base of C than $2\delta/4$. There is a single-valued surface S_2 above the base of C and S_1 such that (1) each point of $S_2 \cdot C$ is $2\delta/4$ units above the base of C and (2) S_2 does not intersect the image under T_2 of an element of G_{n_2} that intersects S_1. To ensure that (2) will be satisfied, we need only take S_2 sufficiently close to the portion of C that lies above S_2.

In a second application of Theorem 4 we let C, G_1, K, and R in the statement of Theorem 4 be C, the collection of images under T_2 of the elements of G_{n_2}, S_2, and the set of points of C which are not nearer the base of C than $3\delta/4$. We find that there is an integer n_3 and a homeomorphism T_3 of E^3 into itself such that (1) T_3 preserves T_2 on the sum of the boundaries of the elements of G_{n_2}, and (2) if g is a bounded element of G_{n_3}, $T_3(\bar{g})$ does not intersect both S_2 and a point of C whose distance is as much as $3\delta/4$ from the base of C. There is a single-valued surface S_3 between S_2 and the top of C such that (1) each point of $S_3 \cdot C$ is $3\delta/4$ units above the base of C and (2) the image under T_3 of no element of G_{n_3} intersects both S_2 and S_3.

Similarly there are mutually exclusive surfaces $S_3, S_4, \cdots, S_j$ ($j\delta/4$ <height of $C \leqq (j+1)\delta/4$), an integer $n_j = n_z$, and a homeomorphism T_j of E^3 into itself such that (1) S_i is single-valued above the base of C, (2) each point of $S_i \cdot C$ is $i\delta/4$ units above the base of C, (3) T_j leaves invariant each boundary point of each element of G_n, and (4) no image under T_j of an element of G_{n_z} intersects two of the surfaces $S_1, S_2, \cdots, S_j$.

There is a homeomorphism T' of E^3 into itself which leaves each point of C invariant, which carries $S_1, S_2, \cdots, S_j$ into surfaces parallel to the base of C and such that the image of a point under T' has the same x and y values as the point has. Then the transformation $T'T_j = T_z$ is one which (1) leaves invariant each point of C, (2) does not move any point of the boundary of the sum of the elements of G_n in the x or y directions, and (3) if p and q are two points of a bounded element of G_{n_z}, the z coordinates of $T_z(p)$ and $T_z(q)$ do not differ by more than $\delta/2$.

Similarly, we find that there is a homeomorphism T_y of E^3 into itself and an integer n_y greater than n_z such that (1) T_yT_z leaves invariant each point of C plus its exterior, (2) if p is a point of the boundary of an element of G_{n_z}, $T_yT_z(p)$ has the same x and z coordinates as $T_z(p)$, (3) if p and q are two points of the same bounded element of G_{n_y} the y coordinates of $T_yT_z(p)$ and $T_yT_z(q)$ do not differ by more than $\delta/2$. Also, there is an integer n_x greater than n_y and a homeomorphism T_x of E^3 into itself such that (1) $T_xT_yT_z$ leaves invariant each point of C plus its interior, (2) if p is a point of the boundary of an element of G_{n_y}, $T_xT_yT_z(p)$ has the same y and z coordinates as $T_yT_z(p)$, and (3) if p and q are two points of the same bounded element of G_{n_x}, then the x coordinates of $T_xT_yT_z(p)$ and $T_xT_yT_z(q)$ do not differ by more than $\delta/2$.

If p and q are two points of a bounded element of G_{n_x}, the y coordinates of $T_xT_yT_z(p)$ and $T_xT_yT_z(q)$ do not differ by more than $\delta/2$ because p and q belong to an element of G_{n_y}, the y coordinates of $T_yT_z(p)$ and $T_yT_z(q)$ do not differ by as much as $\delta/2$, and T_x does not alter the length in the y direction of an element of G_{n_y}. Furthermore, we find that if p and q are two points of a bounded element of G_{n_x}, the z coordinates of $T_xT_yT_z(p)$ and $T_xT_yT_z(q)$ do not differ by more than $\delta/2$. Hence, the image under $T_xT_yT_z$ of each bounded element of G_{n_x} has a diameter of less than δ. Then n_x is the integer and $T_xT_yT_z$ the transformation required in Theorem 5.

THEOREM 6. *The compact continua M_1 and M_2 are homeomorphic if there is a decreasing sequence of partitionings G_{i1}, G_{i2}, $\cdots$ $(i=1, 2)$ for M_i and a 1-1 correspondence between the elements of G_{1j} and G_{2j} such that* (1) *two elements of G_{1j} have a boundary point in common if and only if the corresponding elements of G_{2j} have a boundary point in common and* (2) *corresponding elements of G_{1j+1} and G_{2j+1} are subsets of corresponding elements of G_{1j} and G_{2j}.*

Proof of sufficiency in Theorem 1. Suppose that the ordering of G_1 mentioned in condition (1.4) is $g_1, g_2, \cdots, g_n$. We first show that $\bar{g}_1$ is topologically equivalent to a 3-cell.

Let C be the surface of a cube in E^3 and T_1 be a homeomorphism carrying $F(g_1)$ into C. We find from Theorem 3 that there is a sequence of partitionings $H_1, H_2, \cdots$ of E^3 and a sequence of homeomorphisms $T_1, T_2, \cdots$ satisfying the following:

(1.5) The boundary of each element of H_i is a tame simple surface.

(1.6) H_{i+1} is a refinement of H_i.

(1.7) One element of H_i is the exterior of C.

(1.8) There is a 1-1 correspondence between the bounded elements of H_i and the elements of G_i in g_1 such that

(a) two bounded elements of H_i are adjacent if and only if the corresponding element of G_i in g_1 are adjacent and

(b) corresponding elements of H_{i+1} and G_{i+1} are subsets of corresponding elements of H_i and G_i.

(1.9) T_i is a homeomorphism of the boundary of the sum of the elements of G_i in g_1 into the boundary of the sum of the elements of H_i such that T_i preserves T_{i-1} on the boundary of the sum of the elements of G_{i-1} in g_1 and T_i carries the boundary of an element of G_i in g_1 into the boundary of the corresponding element of H_i.

Although the diameters of the bounded elements of H_i may not approach 0 as i increases without limit, $H_1, H_2, \cdots$ will satisfy a condition like (5.1) because $G_1, G_2, \cdots$ is a decreasing sequence of partitionings and T_{i+n} preserves T_i on the sum of the boundaries of the elements of G_i in $\bar{g}_1$.

We find from Theorem 5 that there is an integer n_1 and a homeomorphism F_1 of E^3 into itself such that F_1 leaves each point of the exterior of C invariant and F_1 carries each bounded element of H_{n_1} into a set of diameter less than 1/2. We let K_1 be the partitioning of E^3 whose elements are the images of the elements of H_{n_1} under F_1.

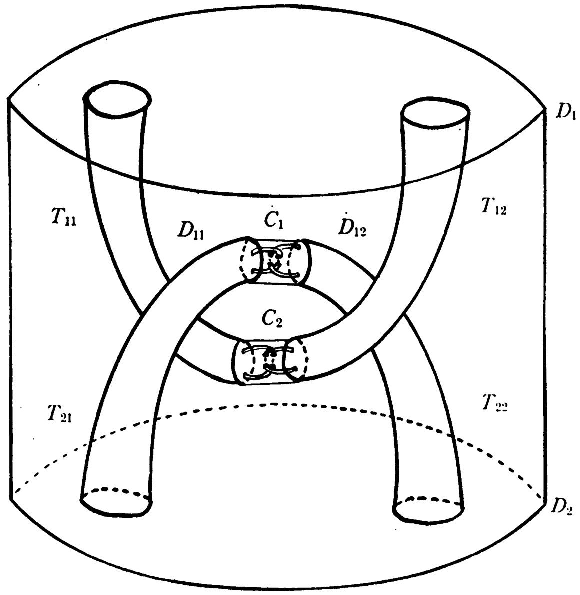

FIG. 4

An application of Theorem 5 to the elements of K_1 reveals that there is an integer n_2 and a homeomorphism F_2 of E^3 into itself such that (1) if p is a boundary point of an element of K_1, p is invariant under F_2, and (2) if h is a bounded element of H_{n_2}, the diameter of the image of h under F_2F_1 is less than 1/4. We denote the collection of images of elements of H_{n_2} under F_2F_1 by K_2. If T_{n_1} is the transformation mentioned in (1.9), we note that $F_1T_{n_1}$ is a homeomorphism of the boundary of the sum of the elements of G_{n_1} in g_1 into the boundary of the sum of the elements of K_1 and that $F_2F_1T_{n_2}$ is a homeomorphism of the boundary of the sum of the elements of G_{n_2} in g_1 into the boundary of the sum of the elements of K_2. Furthermore, $F_2F_1T_{n_2}$ preserves $F_1T_{n_1}$ on these boundaries.

Similarly, we find that there are integers n_3, n_4, $\cdots$ and partitionings K_3, K_4, $\cdots$ of E^3 satisfying conditions like conditions (1.5), (1.6), (1.7), and (1.8) satisfied by H_3, H_4, $\cdots$ and such that each bounded element of K_i is of diameter less than $1/2^i$.

We find from Theorem 6 that these conditions imply that $\bar{g}_1$ and C plus its interior are homeomorphic. A similar line of argument shows that $\bar{g}_2 + \bar{g}_3 + \cdots + \bar{g}_n$ is also homeomorphic to C plus its interior. Since each of these correspondences relates $F(g_1)$ to C, S is topologicallyequivalent to S^3.

Possible strengthening of Theorem 1. It may be noted that in neither Theorem 3 nor in Theorem 5 did we use the full strength of condition (1.2). Hence, at the expense of conciseness, a strengthening of the sufficiency part of Theorem 1 could be made here. For purposes of application in Theorem 1, condition (3.2) deals with $F(g')\cdot F(g'')$ where g', g'' are elements of G_{i+1} in the same element of G_i; condition (4.2) deals with $F(g')\cdot F(g'')$ where g', g'' are elements of different G_i's and g' contains g''.

Consider the right cylinder C with bases D_1 and D_2. The interiors of two mutually exclusive discs in D_i $(i=1, 2)$ are replaced by the surfaces of tubes T_{i1} and T_{i2} and discs D_{i1} and D_{i2} as shown in Figure 4 where D_{i1} and D_{i2} are the bases of a right circular cylinder C_i and $D_1+T_{11}+C_1+T_{12}$ is "hooked" to $D_2+T_{21}+C_2+T_{12}$ as shown. Discs in the bases of the cylinders C_1 and C_2 are replaced by surfaces of tubes and discs as before. If the process is continued and the limit taken, a wild simple surface M is obtained. If g is an element of G_1 and condition (1.2) is replaced by one like (3.2), M might be the image of $F(g)$ in a homeomorphism of S into S^3 if S is topologically equivalent to S^3 and the elements of G_2, G_3, $\cdots$ in g are fitted together as suggested after Theorem 4 in Figure 3. In fact, if M_1 and M_2 are two Alexander "horned spheres"(4) in S^3, T is a homeomorphism of M_1 into M_2, and U_i $(i=1, 2)$ is one of the complementary domains of M_i, then S may be topologically equivalent to $U_1+M_1+U_2$ where it is understood that a point p of M_1 is a limit point of a subset V of U_2 if and only if $T(p)$ is a limit point of V(5).

Condition (1.3) prevents S from being the sum of three 3-cells such that the common part of any two is the boundary of each. However, Theorem 1 would be somewhat stronger if the expression "1-dimensional" could be replaced by "0-dimensional or 1-dimensional."

Condition (1.4) rules out many 3-manifolds (as well as some spaces that are not 3-manifolds). It imposes a type of sequential unicoherence. However, unicoherence alone does not appear to be strong enough because projective 3-space has a decreasing sequence of regular partitionings satisfying conditions (1.1), (1.2), and (1.3). One might wonder if simple connectedness might be substituted for condition (1.4).

Since E^3 is topologically equivalent to S^3 minus a point, the following is immediately obtained from Theorem 1.

THEOREM 7. *A metric space S' is topologically equivalent to E^3 provided it has a sequence of regular partitionings G_1, G_2, $\cdots$ satisfying conditions* (1.1), (1.2), (1.3), *and* (1.4) *as well as the following*: *G_{i+1} is a refinement of G_i; exactly one element of G_i fails to have a compact closure; for each point p there is an*

(4) J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 8–10.

(5) R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, New York, 1949, Problem 4.6, p. 383.

integer $n(p)$ such that p does not belong to the closure of this exceptional element of $G_{n(p)}$; the diameters of the elements of G_i other than this exceptional element approach 0 as i increases without limit.

Characterization of a simple closed curve. Since a 1-manifold without boundary is a simple closed curve, a simple closed curve is characterized by its local properties. Hence the following characterization of a simple closed curve is shorter than the corresponding characterizations of a simple surface and a 3-sphere.

THEOREM 8. *A necessary and sufficient condition that S be a simple closed curve is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties:*

(8.1) *The boundary of each element of G_i is a pair of distinct points.*

(8.2) *No three elements of G_i have a boundary point in common.*

In fact, each decreasing sequence of partitionings $H_1, H_2, \cdots$ of a simple closed curve has properties (8.1) and (8.2) if H_1 has more than one element. Condition (8.2) could be replaced by the condition that the partitionings are brick partitionings.

Characterization of a simple surface. When a simple closed curve is partitioned into more than two elements, the boundary of each element of the partitioning is a pair of points. However, when a simple surface is partitioned there is a bigger variety of possible boundaries. They may either be simple closed curves or the sum of a finite number of simple closed curves.

THEOREM 9. *A necessary and sufficient condition that S be a simple surface is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties:*

(9.1) *The boundary of each element of G_i is a simple closed curve.*

(9.2) *The intersection of the boundaries of 3 elements of G_i contains no arc.*

(9.3) *If g is an element of G_{i-1} ($g = S$ if $i=1$) the elements of G_i in g may be ordered $g_1, g_2, \cdots, g_n$ so that $F(g_j)$ $[j=1$ (if $i>1$), $2, \cdots, n]$ intersects $F(g)+G(g_1)+ \cdots +F(g_{j-1})$ in a nondegenerate connected set.*

Indication of proof. Conditions (9.1), (9.2), and (9.3) correspond to conditions (1.1), (1.3), and (1.4) of Theorem 1. The word "nondegenerate" in condition (9.3) replaces the equivalent of condition (1.2) of Theorem 1. This condition cannot be deleted entirely or else S could be the sum of two simple surfaces with only one point in common.

To prove Theorem 9, prove a theorem resembling Theorem 3 showing that there are a sequence of partitionings $H_1, H_2, \cdots$ of S^2 such that H_{i+1} refines H_i and a sequence of homeomorphisms $T_1, T_2, \cdots$ such that T_i carries the boundary of the sum of the elements of G_i into the boundary of the sum of the elements of H_i and T_{i+1} preserves T_i. Next prove a result like Theorem 5 showing that there is no loss of generality in supposing that the maximum of the diameters of the elements of H_i approaches 0 as i increases without limit. Finally, use Theorem 6 to establish a homeomorphism between S and S^2.

The following result throws some light on the role of condition (9.3). It follows from the fact that for continua H, K in S^2, $S^2-(H+K)$ is not connected if $H \cdot K$ is not connected.

THEOREM 10. *Suppose G_1 and G_2 are two partitionings of S^2, G_2 refines G_1, the boundaries of the elements of G_1 and G_2 are connected, and g is either S^2 or an element of G_1 whose boundary is nondegenerate. Then the elements of G_2 in g may be ordered $g_1, g_2, \cdots, g_n$ such that $F(g_j)$ intersects $F(g)+F(g_1)+ \cdots + F(g_{j-1})$ in a nondegenerate connected set.*

It would be interesting to know what the corresponding result is for S^3. In particular, if $G_1, G_2, \cdots$ are partitionings of S^3 satisfying conditions (1.1), (1.2), and (1.3), would they automatically satisfy condition (1.4)?

UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VA.
UNIVERSITY OF WISCONSIN,
MADISON, WIS.